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Freezing of Dissociation and Recombination in Supersonic Nozzle Flows: Part II. More General Analysis and Numerical Example

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The special analysis of Pt. I on the nonequilibrium nozzle flow of an ideal dissociating gas is generalized to encompass the entire range of the reference dissociation level in the reaction-dominated regime. The hypersonic restriction on the analysis of Pt. I is also relaxed, although the equilibrium-throat condition is still retained. The results show that the assessment of the equilibrium-recombination and the sudden-freezing models in Pt. I hold in general for the reaction-dominated regime. A numerical example is presented to compare the results given by the analysis of Pt. I and the exact numerical solution. Good agreement between them is obtained.

I. Introduction

In Pt. I of this study,[1] the nonequilibrium nozzle flow of an ideal dissociating gas has been discussed in general. It is pointed out that the problems fall into two main regimes depending on the degree of interaction between chemistry and thermodynamics. In the strongly coupled, reaction-dominated regime the flow from an initial equilibrium to a frozen or recombination-controlled state encompasses four regions: 1) equilibrium, 2) incipient-transition, 3) rapid-transition, and 4) recombination. A reference point, similar to Bray's freezing point,[2] is chosen to coincide with the singular point of the incipient-transition solution. A hypersonic, slightly dissociated case in this regime has been analyzed. Its simple, explicit solution was used to assess the domain of validity and the errors of the equilibrium-recombination and the sudden-freezing models.

In this paper, the slight-dissociation restriction on the analysis of Pt. I will be removed. The entire range of α_* (or K)‡ in the reaction-dominated regime can be inclusively

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‡ Notations used in this paper are the same as those in Ref. 1, unless otherwise defined

studied by considering the two extreme cases: $K = 0(1)$ or $\alpha_* = 0(\epsilon)$ and $K = 0(1/\epsilon)$ or $\alpha_* = 0(1)$. All intermediate cases can be obtained by either letting K in the first case become large or letting α_* in the second case become small. In carrying out this study, the hypersonic restriction on the analysis of Pt. I will also be relaxed by allowing M_*^2 to be smaller than $0(1/\epsilon)$, although still large enough [larger than $0(\ln\epsilon)$] that the equilibrium-throat assumption holds. The results will show agreement with that of Pt. I when they are specialized to the hypersonic, slightly dissociated case. More importantly, the results will show that the assessment of the equilibrium-recombination and the sudden-freezing models in Pt. I holds in general for the reaction-dominated regime.

To further enhance confidence in our analysis, a numerical example will be presented. It deals with a hyperboloidal nozzle with $\epsilon = 0.0535$, $\alpha_* = 0.320$, and $M_*^2 = 27.3$. The results given by the asymptotic solution of Pt. I are compared with those given by Emanuel's numerical solution to the full equations[3] [Eqs. (2.1–2.6) of Ref. 1]. The agreement is unexpectedly good considering that α_* is not very small in this case.

2. The Case of $K = 0(1)$

This case is different from the case analyzed in Pt. I only in the relaxation of the hypersonic restriction. Equations (2.10b,c,d) and (2.12) of Pt. I are the governing equations. Matched asymptotic expansions for small ϵ with K and M_*^2 fixed will be developed for each of the four regions, in a manner similar to that of Pt. I.

2.1 Equilibrium Region

In the region upstream of the reference point where $\theta < 1$, the convective term in the rate equation can be neglected, with an exponentially small error in ϵ. However, unlike the case in Pt. I, the real error involved here is of the type $1/\exp|0(M_*^2)|$, which is invoked by the equilibrium-throat assumption and is more severe than the error of $1/\exp|0(1/\epsilon)|$. These two types of error become of the same order only in the hypersonic case, $M_*^2 = 0(1/\epsilon)$. Now the resulting equilibrium-flow solution is assumed to be known, and only its behavior near $\theta = 1$ is needed for matching with the incipient-transition solution. This behavior can be shown to be

$$\begin{aligned} \tilde{\alpha} &= \alpha_0 + \epsilon\alpha_1 + 0(\epsilon^2) \\ \theta &= 1 + \epsilon\theta_1 + 0(\epsilon^2) \\ \tilde{\rho} &= \rho_0 + \epsilon\rho_1 + 0(\epsilon^2) \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \alpha_0 &= 1 + (\ln\rho_0)/K \\ \rho_0^2[1 - (2\ln\rho_0)/M_*^2] &= 1/\sigma^2 \\ \theta_1 &= -\ln[\rho_0\{1 + (\ln\rho_0)/K\}^2] \\ K\alpha_1 - (M_*^2/\rho_0^2\sigma^2)(\rho_1/\rho_0) &= 4\theta_1 - K(\alpha_0 - 1) \\ K\alpha_1 - (\rho_1/\rho_0) &= 3\theta_1 + K\ln\rho_0 + (\ln\rho_0)^2 + 2K(\alpha_0\ln\alpha_0 - \alpha_0 + 1) \end{aligned} \tag{2.2}$$

2.2 Incipient-Transition Region

This region is characterized by the proximity of θ to unity. The asymptotic expansions here should match with the equilibrium-flow solution, and hence have the same form as those in Eq. (2.1). Substituting the asymptotic expansions into the governing equations yields the following result:

$$\begin{aligned} \alpha_0 &= 1 + (\ln\rho_0)/K \\ \rho_0^2[1 - (2\ln\rho_0)/M_*^2] &= 1/\sigma^2 \\ \theta_1 &= -\ln[\rho_0\{1 + (\ln\rho_0)/K\}^2 - QM_*^2g/\rho_0^4\sigma^4(M_*^2 - 1 - 2\ln\rho_0)] \\ K\alpha_1 - (M_*^2/\rho_0^2\sigma^2)(\rho_1/\rho_0) &= 4\theta_1 - K(\alpha_0 - 1) \\ K\alpha_1 - \frac{\rho_1}{\rho_0} &= 3\theta_1 + K\ln\rho_0 + \frac{(\ln\rho_0)^2}{2} - \int_1^{\rho_0}\frac{\theta_1 d\rho_0}{\rho_0} + KB_0 \end{aligned} \tag{2.3}$$

It is clear that the integration constant of α_0 has been determined to be unity by comparing with the α_0 expression in Eq. (2.2).

Unlike the case in Pt. I, this result cannot be formulated explicitly in terms of σ, but is expressed instead in terms of the variable ρ_0. It is noted that there are two particular values of ρ_0 of special physical interest, corresponding to the reservoir and the throat conditions, respectively. The second of Eqs. (2.3) shows that the reservoir ($\tilde{u} = 1/\rho_0\sigma = 0$) corresponds to $\rho_0 = \exp(M_*^2/2)$ while the throat $d\sigma = 0$ occurs at $\rho_0 = \exp[(M_*^2 - 1)/2]$ or $\sigma = M_*\exp[-(M_*^2 - 1)/2]$. Since the throat appears as a singularity, its neighborhood must be treated by a separate set of asymptotic expansions. This may be avoided by carrying out the matching with the equilibrium solution sufficiently downstream of the critical point, although the incipient-transition region may well extend into the subcritical part of the nozzle. [The region terminates upstream at $\rho_0 = \exp|0(1/\epsilon)|$]. Thus, the present matching procedure and its consequent error are different from those in Pt. I.

Comparing Eqs. (2.2) and (2.3), we see that the differences between them are the Q-terms and the undetermined integration constant B_0 in Eq. (2.3). From the Bernoulli equation, it can be shown that the local Mach number $M^2 \equiv u^2/R_2T_* = M_*^2 - 2\ln\rho_0 + 0(\epsilon)$ or $\rho_0 = \exp|0(M_*^2 - M^2)|$ in the incipient-transition region. Let the matching be carried out downstream of the critical point where $M^2 = 0(1)$. Making use of assumption 3 in Sec. 2 of Pt. I, the matching is accomplished by taking

$$B_0 = -\frac{1}{K}\int_1^\infty \ln\left[1 - \frac{QM_*^2g}{\alpha_0^2\rho_0^5\sigma^4(M_*^2 - 1 - 2\ln\rho_0)}\right]\frac{d\rho_0}{\rho_0} \tag{2.3a}$$

with an error equal to $1/\exp|0(M_*^2)|$. Furthermore, it can be inferred from the self-consistency that the error involved in the equilibrium-throat assumption is of the same order.

At the downstream end of the incipient-transition region, another singularity emerges where

$$\rho_0[1 + (\ln\rho_0)/K]^2 - QM_*^2g/\rho_0^4\sigma^4(M_*^2 - 1 - 2\ln\rho_0) = 0 \tag{2.4}$$

To insure that this point coincides with the transition point $\sigma = 1$, Q is chosen to be $Q = 1 - 1/M_*^2$, which is different but consistent with that given in Pt. I. Furthermore, an examination shows that, in the range of lower M_*, corresponding to this Q, $P = 1 + (\epsilon)$. Now, as in Pt. I, it can be shown that the region of nonuniformity is $|\sigma - 1| = 0(\epsilon^{1/2})$ and the appropriate independent variable to be used there should be

$$\eta = (\sigma - 1)/\epsilon^{1/2} \tag{2.5}$$

To provide the matching conditions, the asymptotic behavior of the incipient-transition solution near $\sigma = 1$ is expressed in terms of η as follows:

$$\rho_0 = 1 - \epsilon^{1/2}\frac{M_*^2\eta}{(M_*^2-1)} + \epsilon\frac{M_*^2(2M_*^4-3M_*^2+3)}{2(M_*^2-1)^3}\eta^2 + \ldots$$

$$\alpha_0 = 1 - \epsilon^{1/2}\frac{M_*^2\eta}{K(M_*^2-1)} + \frac{\epsilon M_*^2(M_*^4-2M_*^2+3)}{2K(M_*^2-1)^3}\eta^2 + \ldots$$

$$\theta_1 = -\ln|3\bar{\beta}\epsilon^{1/2}\eta| + \ldots \quad (2.6)$$

$$\rho_1 = \frac{1}{(M_*^2-1)}[\ln|3\bar{\beta}\epsilon^{1/2}\eta| + KB_0] + \ldots$$

$$\alpha_1 = \frac{1}{K(M_*^2-1)}[(4-3M_*^2)\ln|3\bar{\beta}\epsilon^{1/2}\eta| + M_*^2KB_0] + \ldots$$

where

$$\bar{\beta} \equiv \frac{1}{3}\left\{\frac{M_*^2}{(M_*^2-1)}\left(5 + \frac{2}{K} - \frac{2}{M_*^2-1}\right) + g_*' - 4\right\} > 0$$

The remainders in Eqs. (2.6) are proportional to $\epsilon^{3/2}$ for ρ_0 and α_0, and to $\epsilon^{1/2}$ for θ_1, ρ_1 and α_1. According to assumption 6 in Sec. 2 of Pt. I, $\bar{\beta}$ is positive.

2.3 Rapid-Transition Region

In view of Eq. (2.6), the asymptotic expansions for this region are assumed to be

$$\begin{aligned}\tilde{\alpha} &= 1 + \epsilon^{1/2}\bar{\alpha}_1 + \epsilon\ln\epsilon\bar{\alpha}_{21} + \epsilon\bar{\alpha}_{22} + 0(\epsilon^{3/2})\\ \tilde{\rho} &= 1 + \epsilon^{1/2}\bar{\rho}_1 + \epsilon\ln\epsilon\bar{\rho}_{21} + \epsilon\bar{\rho}_{22} + 0(\epsilon^{3/2}) \quad (2.7)\\ \theta &= 1 + \epsilon\ln(\bar{\theta}_1/\epsilon^{1/2}) + 0(\epsilon^{3/2})\end{aligned}$$

where the barred quantities are functions of η. Substituting these into the governing equations yields the following result:

$$\bar{\rho}_1 = K\bar{\alpha}_1 = \frac{-M_*^2\eta}{(M_*^2-1)} \qquad \bar{\theta}_1 = \frac{a}{3}e^{a\bar{\beta}\eta^2/2}\int_{-\infty}^{\eta}e^{-a\bar{\beta}\tau^2/2}d\tau$$

$$\bar{\rho}_{21} = 1/2(M_*^2-1) \qquad \bar{\alpha}_{21} = (4-3M_*^2)/2K(M_*^2-1) \quad (2.8)$$

$$\bar{\rho}_{22} = \frac{1}{(M_*^2-1)}\left\{KB_0 + \frac{M_*^2(2M_*^4-3M_*^2+3)}{2(M_*^2-1)^2}\eta^2 - \ln\left|\frac{a}{3}e^{a\bar{\beta}\eta^2/2}\int_{-\infty}^{\eta}e^{-a\bar{\beta}\tau^2/2}d\tau\right|\right\}$$

$$\bar{\alpha}_{22} = \frac{1}{K(M_*^2-1)}\left\{M_*^2KB_0 + \frac{M_*^2(M_*^4-2M_*^2+3)}{2(M_*^2-1)^2}\eta^2 + (3M_*^2-4)\times\ln\left|\frac{a}{3}e^{a\bar{\beta}\eta^2/2}\int_{-\infty}^{\eta}e^{-a\bar{\beta}\tau^2/2}d\tau\right|\right\}$$

where

$$a \equiv 3M_*^2/(3M_*^2-4)$$

and the integration constants have been determined by matching with Eq. (2.6).

The expansions in Eq. (2.7) cease to be valid sufficiently downstream of the transition point where θ becomes considerably larger than unity and the recombination effect becomes dominant. In the recombination region σ will be used again in place of η. The asymptotic behavior of the rapid-transition solution near the recombination region can be obtained by letting η approach positive infinity and expressing the results in terms of $(\sigma - 1)$:

$$\tilde{\alpha} = 1 - \frac{M_*^2(\sigma-1)}{K(M_*^2-1)} + \left[\frac{M_*^2(M_*^4-2M_*^2+3)}{2K(M_*^2-1)^3} + \frac{(3M_*^2-4)a\bar{\beta}}{2K(M_*^2-1)}\right](\sigma-1)^2 + \epsilon\frac{(3M_*^2-4)}{2K(M_*^2-1)}\times\ln\left(\frac{2\pi a}{9\bar{\beta}\epsilon}\right) + \epsilon M_*^2B_0/(M_*^2-1) + \ldots$$

$$(2.9)$$

$$\tilde{\rho} = 1 - \frac{M_*^2(\sigma-1)}{(M_*^2-1)} + \left[\frac{M_*^2(2M_*^4-3M_*^2+3)}{2(M_*^2-1)^3} - \frac{a\bar{\beta}}{2(M_*^2-1)}\right](\sigma-1)^2 - \frac{\epsilon}{2(M_*^2-1)}\ln\left(\frac{2\pi a}{9\bar{\beta}\epsilon}\right) + \epsilon KB_0/(M_*^2-1) + \ldots$$

$$\theta = 1 + \frac{a\bar{\beta}(\sigma-1)^2}{2} + \frac{\epsilon}{2}\ln\left(\frac{2\pi a}{9\bar{\beta}\epsilon}\right) + \ldots$$

where the remainders are all proportional to $\epsilon^{3/2}$. Equations (2.9) will serve to determine the integration constants in the next region.

2.4 Recombination Region

With the guidance of Eq. (2.9), asymptotic expansions in terms of ϵ can be assumed for this region just as for the transitional regions. However, such expansions do not yield explicit results in general and lead to nonuniformities far downstream in certain cases, as discussed in Pt. I. The solution for this region will simply be the governing equations put in the integral-equation form without the dissociation-rate term, which is exponentially small when $\theta > 1$. They are

$$\frac{1}{\tilde{\alpha}} = 1 - \frac{\epsilon M_*^2B_0}{(M_*^2-1)} - \frac{\epsilon}{K}\frac{(3M_*^2-4)}{2(M_*^2-1)}\ln\left(\frac{2\pi a}{9\bar{\beta}\epsilon}\right) + \frac{M_*^2}{K(M_*^2-1)}\int_1\frac{\theta^s\tilde{\rho}^3\sigma d\sigma}{g} + 0(\epsilon^{3/2})$$

$$\frac{1}{\theta\tilde{\rho}^{(1+\alpha_\infty)/3}} = 1 - \frac{\epsilon KB_0}{3(M_*^2-1)} - \epsilon\frac{(3M_*^2-4)}{6(M_*^2-1)}\times\ln\left(\frac{2\pi a}{9\bar{\beta}\epsilon}\right) + \frac{M_*^2}{3(M_*^2-1)}\int_1\frac{\theta^s\tilde{\rho}^3\tilde{\alpha}^2\sigma d\sigma}{g\tilde{\rho}^{(1+\alpha_\infty)/3}} + \frac{\epsilon K}{3}\int_1\frac{(\tilde{\alpha}-\tilde{\alpha}_\infty)d\tilde{\rho}}{\theta\tilde{\rho}^{1+(1+\alpha_\infty)/3}} + 0(\epsilon^{3/2}) \quad (2.10)$$

$$(M_*^2/2)(1/\tilde{\rho}^2\sigma^2 - 1) = K(1-\tilde{\alpha}) + 4(1-1/\theta) + \epsilon K(1-\tilde{\alpha}/\theta)$$

where the integration constants are determined by matching with Eq. (2.9).

It is easily shown that Eqs. (2.1–2.10) reduce to their counterparts in Pt. I when M_*^2 is taken to be $0(1/\epsilon)$. The present results verify that the equilibrium-recombination model is a uniformly valid approximation for all flow variables with an error of $0(\epsilon\ln\epsilon)$, just as concluded in Pt. I, except that the transition point should be determined by $Q = 1 - 1/M_*^2$ in the lower M_* cases. Moreover, it is again established that $K(1-\tilde{\alpha}) = 0(1)$ if the flow freezes far downstream (the conditions for freezing have been derived in Pt. I without restrictions on α_* and M_*). The assessment on the sudden-freezing model, as made in Pt. I, remains unchanged for the present case.

3. The Case of $\alpha_* = 0(1)$

In this case, the dissociation fraction near the transition point is no longer small and K is of the order $(1/\epsilon)$. The governing equations will be Eqs. (2.10b,c,d) and (2.12) of Pt. I with K replaced by α_*/ϵ. Matched asymptotic expansions for small ϵ with α_* and M_* fixed can be developed for each of the four regions in the same manner as in the previous section.

3.1 Equilibrium Region

The asymptotic behavior of the equilibrium solution near $\theta = 1$ can be shown to be

$$\tilde{\alpha} = 1 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + 0(\epsilon^3)$$
$$\tilde{\rho} = \rho_0 + \epsilon\rho_1 + 0(\epsilon^2) \qquad \theta = 1 + \epsilon\theta_1 + 0(\epsilon^2) \tag{3.1}$$

where

$$\alpha_1 = (1 + \alpha_*)\,(\ln\rho_0)/\alpha_*$$
$$\rho_0{}^2[1 - (1 + \alpha_*)(2\ln\rho_0)/M_*{}^2] = 1/\sigma^2$$
$$\theta_1 = -\ln\rho_0 \tag{3.2}$$
$$\alpha_*\alpha_2 - (M_*{}^2/\rho_0{}^2\sigma^2)(\rho_1/\rho_0) = 4\theta_1 + \alpha_*(\theta_1 - \alpha_1)$$
$$\alpha_*\alpha_2 - (1 + \alpha_*)(\rho_1/\rho_0) = 3\theta_1 + (1 + \alpha_*)(\ln\rho_0)^2$$

Note that the expansion for $\tilde{\alpha}$ is now different from its counterpart in the case of $K = 0(1)$.

3.2 Incipient-Transition Region

The asymptotic expansions for this region are assumed to be of the same form as Eq. (3.1), and the results are as follows:

$$\alpha_1 = (1 + \alpha_*)(\ln\rho_0)/\alpha_*$$
$$\rho_0{}^2[1 - (1 + \alpha_*)(2\ln\rho_0)/M_*{}^2] = 1/\sigma^2$$
$$\theta_1 = -\ln[\rho_0 - (1 + \alpha_*)QM_*{}^2g/\rho_0{}^4\sigma^4\{M_*{}^2 - (1 + \alpha_*)(1 + 2\ln\rho_0)\}] \tag{3.3}$$
$$\alpha_*\alpha_2 - (M_*{}^2/\rho_0{}^2\sigma^2)(\rho_1/\rho_0) = 4\theta_1 + \alpha_*(\theta_1 - \alpha_1)$$
$$\alpha_*\alpha_2 - (1 + \alpha_*)\left(\frac{\rho_1}{\rho_0}\right) = 3\theta_1 + \frac{(1 + \alpha_*)(\ln\rho_0)^2}{2} - (1 + \alpha_*)\int_1^{\rho_0}\frac{\theta_1 d\rho_0}{\rho_0} + \alpha_* B_0$$

and

$$B_0 = -\frac{(1 + \alpha_*)}{\alpha_*}\int_1^\infty \ln\left[1 - \frac{(1 + \alpha_*)QM_*{}^2g}{\rho_0{}^5\sigma^4\{M_*{}^2 - (1 + \alpha_*)(1 + 2\ln\rho_0)\}}\right]\frac{d\rho_0}{\rho_0} \tag{3.3a}$$

Now the downstream singularity of this region is located at

$$\rho_0 - (1 + \alpha_*)QM_*{}^2g/\rho_0{}^4\sigma^4\{M_*{}^2 - (1 + \alpha_*) \times (1 + 2\ln\rho_0)\} = 0 \tag{3.4}$$

and is coincident with the transition point $\sigma = 1$ if Q is chosen to be

$$Q = [M_*{}^2 - (1 + \alpha_*)]/M_*{}^2(1 + \alpha_*)$$

which, in this range of M_* and α_*, gives again $P = 1 + 0(\epsilon)$. Also, since this Q contains the previously determined values as special cases, it is the most general form to be used in the reaction-dominated regime. As before, the independent variable to be used in the rapid-transition region is $\eta = (\sigma - 1)/\epsilon^{1/2}$. The asymptotic behavior of the incipient-transition solution when expressed in terms of η is

$$\rho_0 = 1 - \epsilon^{1/2}\frac{M_*{}^2\eta}{[M_*{}^2 - (1 + \alpha_*)]} + \epsilon\frac{M_*{}^2[2M_*{}^4 - 3M_*{}^2(1 + \alpha_*) + 3(1 + \alpha_*)^2]\eta^2}{2[M_*{}^2 - (1 + \alpha_*)]^3} + \ldots$$

$$\alpha_1 = -\epsilon^{1/2}\frac{M_*{}^2(1 + \alpha_*)\eta}{\alpha_*[M_*{}^2 - (1 + \alpha_*)]} + \epsilon\frac{M_*{}^2(1 + \alpha_*)[M_*{}^4 - 2M_*{}^2(1 + \alpha_*) + 3(1 + \alpha_*)^2]\eta^2}{2\alpha_*[M_*{}^2 - (1 + \alpha_*)]^3} + \ldots$$

$$\theta_1 = -\ln|3\bar{\bar{\beta}}\epsilon^{1/2}\eta| + \ldots \tag{3.5}$$

$$\rho_1 = \frac{1}{[M_*{}^2 - (1 + \alpha_*)]} \times [(1 + \alpha_*)\ln|3\bar{\bar{\beta}}\epsilon^{1/2}\eta| + \alpha_* B_0] + \ldots$$

$$\alpha_2 = \left\{\frac{(1 + \alpha_*)^2}{\alpha_*[M_*{}^2 - (1 + \alpha_*)]} - \frac{3}{\alpha_*}\right\} \times \ln|3\bar{\bar{\beta}}\epsilon^{1/2}\eta| + \frac{M_*{}^2 B_0}{[M_*{}^2 - (1 + \alpha_*)]} + \ldots$$

where

$$\bar{\bar{\beta}} = \frac{1}{3}\left\{\frac{5M_*{}^2}{[M_*{}^2 - (1 + \alpha_*)]} - \frac{2M_*{}^2(1 + \alpha_*)}{[M_*{}^2 - (1 + \alpha_*)]^2} + g_*{}' - 4\right\} > 0$$

The remainders in Eq. (3.5) are again proportional to $\epsilon^{3/2}$ for ρ_0 and α_1, and to $\epsilon^{1/2}$ for θ_1, ρ_1, and α_2.

3.3 Rapid-Transition Region

In view of Eq. (3.5), the asymptotic expansions for this region are assumed to be

$$\tilde{\alpha} = 1 + \epsilon^{3/2}\bar{\alpha}_1 + \epsilon^2\ln\epsilon\bar{\alpha}_{21} + \epsilon^2\bar{\alpha}_{22} + 0(\epsilon^{5/2})$$
$$\tilde{\rho} = 1 + \epsilon^{1/2}\bar{\rho}_1 + \epsilon\ln\epsilon\bar{\rho}_{21} + \epsilon\bar{\rho}_{22} + 0(\epsilon^{3/2}) \tag{3.6}$$
$$\theta = 1 + \epsilon\ln(\bar{\theta}_1/\epsilon^{1/2}) + 0(\epsilon^{3/2})$$

Note again that the expansion for $\tilde{\alpha}$ is different from its counterpart in the case of $K = 0(1)$. Substituting Eq. (3.6) into the governing equations yields

$$\bar{\rho}_1 = -M_*{}^2\eta/[M_*{}^2 - (1 + \alpha_*)]$$
$$\bar{\alpha}_1 = -M_*{}^2(1 + \alpha_*)\eta/\alpha_*[M_*{}^2 - (1 + \alpha_*)]$$
$$\bar{\theta}_1 = \frac{\bar{a}}{3}e^{\bar{a}\bar{\bar{\beta}}\eta^2/2}\int_{-\infty}^{\eta} e^{-\bar{a}\bar{\bar{\beta}}\tau^2/2}\,d\tau$$
$$\bar{\rho}_{21} = (1 + \alpha_*)/2[M_*{}^2 - (1 + \alpha_*)]$$
$$\bar{\alpha}_{21} = (1 + \alpha_*)^2/2\alpha_*[M_*{}^2 - (1 + \alpha_*)] - 3/2\alpha_* \tag{3.7}$$
$$\bar{\rho}_{22} = \frac{M_*{}^2[2M_*{}^4 - 3M_*{}^2(1 + \alpha_*) + 3(1 + \alpha_*)^2]\eta^2}{2[M_*{}^2 - (1 + \alpha_*)]^3} + \frac{\alpha_* B_0}{[M_*{}^2 - (1 + \alpha_*)]} - \frac{(1 + \alpha_*)}{[M_*{}^2 - (1 + \alpha_*)]} \times \ln\left|\frac{\bar{a}}{3}e^{\bar{a}\bar{\bar{\beta}}\eta^2/2}\int_{-\infty}^{\eta} e^{-\bar{a}\bar{\bar{\beta}}\tau^2/2}\,d\tau\right|$$

$$\bar{\alpha}_{22} = \frac{M_*{}^2(1 + \alpha_*)[M_*{}^4 - 2M_*{}^2(1 + \alpha_*) + 3(1 + \alpha_*)^2]\eta^2}{2\alpha_*[M_*{}^2 - (1 + \alpha_*)]^3} + \frac{M_*{}^2 B_0}{[M_*{}^2 - (1 + \alpha_*)]} + \left\{\frac{3}{\alpha_*} - \frac{(1 + \alpha_*)^2}{\alpha_*[M_*{}^2 - (1 + \alpha_*)]}\right\} \times \ln\left|\frac{\bar{a}}{3}e^{\bar{a}\bar{\bar{\beta}}\eta^2/2}\int_{-\infty}^{\eta} e^{-\bar{a}\bar{\bar{\beta}}\tau^2/2}\,d\tau\right|$$

where

$$\bar{a} \equiv 3M_*{}^2(1 + \alpha_*)/[3M_*{}^2 - 3(1 + \alpha_*) - (1 + \alpha_*)^2]$$

and the integration constants have been determined by matching with Eq. (3.5). The asymptotic behavior of this solution near the recombination region ($\eta \to \infty$) is

$$\left.\begin{aligned}
\tilde{\alpha} = 1 &- \epsilon \frac{M_*^2(1+\alpha_*)(\sigma-1)}{\alpha_*[M_*^2-(1+\alpha_*)]} + \epsilon \frac{M_*^2(1+\alpha_*)[M_*^4 - 2M_*^2(1+\alpha_*) + 3(1+\alpha_*)^2](\sigma-1)^2}{2\alpha_*[M_*^2-(1+\alpha_*)]^3} + \\
&\frac{\epsilon}{2}\left\{\frac{3}{\alpha_*} - \frac{(1+\alpha_*)^2}{\alpha_*[M_*^2-(1+\alpha_*)]}\right\}\bar{a}\bar{\bar{\beta}}(\sigma-1)^2 + \epsilon^2 \frac{M_*^2 B_0}{[M_*^2-(1+\alpha_*)]} + \frac{\epsilon^2}{2}\left\{\frac{3}{\alpha_*} - \right. \\
&\left.\frac{(1+\alpha_*)^2}{\alpha_*[M_*^2-(1+\alpha_*)]}\right\}\ln\left(\frac{2\pi\bar{a}}{9\bar{\bar{\beta}}\epsilon}\right) + \ldots \\
\tilde{\rho} = 1 &- \frac{M_*^2(\sigma-1)}{[M_*^2-(1+\alpha_*)]} + \frac{M_*^2[2M_*^4 - 3M_*^2(1+\alpha_*) + 3(1+\alpha_*)^2](\sigma-1)^2}{2[M_*^2-(1+\alpha_*)]^3} - \frac{(1+\alpha_*)\bar{a}\bar{\bar{\beta}}(\sigma-1)^2}{[M_*^2-(1+\alpha_*)]} + \\
&\epsilon \frac{\alpha_* B_0}{[M_*^2-(1+\alpha_*)]} - \frac{\epsilon}{2}\frac{(1+\alpha_*)}{[M_*^2-(1+\alpha_*)]}\ln\left(\frac{2\pi\bar{a}}{9\bar{\bar{\beta}}\epsilon}\right) + \ldots \\
\theta = 1 &+ \frac{\bar{a}\bar{\bar{\beta}}(\sigma-1)^2}{2} + \frac{\epsilon}{2}\ln\left(\frac{2\pi\bar{a}}{9\bar{\bar{\beta}}\epsilon}\right) + \ldots
\end{aligned}\right\} \quad (3.8)$$

where the remainders are proportional to $\epsilon^{5/2}$ for $\tilde{\alpha}$, and to $\epsilon^{3/2}$ for $\tilde{\rho}$ and θ.

3.4 Recombination Region

The solution for this region, corresponding to Eq. (2.10), is

$$\begin{aligned}
\frac{1}{\tilde{\alpha}} = 1 &- \frac{\epsilon^2 M_*^2 B_0}{[M_*^2-(1+\alpha_*)]} - \frac{\epsilon^2}{2\alpha_*} \times \\
&\left\{3 - \frac{(1+\alpha_*)^2}{[M_*^2-(1+\alpha_*)]}\right\}\ln\left(\frac{2\pi\bar{a}}{9\bar{\bar{\beta}}\epsilon}\right) + \\
&\frac{\epsilon M_*^2(1+\alpha_*)}{\alpha_*[M_*^2-(1+\alpha_*)]}\int_1^\sigma \frac{\theta^s\tilde{\rho}^3\sigma d\sigma}{g} + 0(\epsilon^{5/2})
\end{aligned} \quad (3.9)$$

$$\begin{aligned}
\frac{1}{\theta\tilde{\rho}^{(1+\alpha_\infty)/3}} = 1 &- \frac{\epsilon\alpha_* B_0}{3[M_*^2-(1+\alpha_*)]} - \\
&\epsilon\frac{[3M_*^2 - 4(1+\alpha_*)]}{6[M_*^2-(1+\alpha_*)]}\ln\left(\frac{2\pi\bar{a}}{9\bar{\bar{\beta}}\epsilon}\right) + \frac{M_*^2(1+\alpha_*)}{3[M_*^2-(1+\alpha_*)]} \times \\
&\int_1^\sigma \frac{\theta^s\tilde{\rho}^3\tilde{\alpha}^2\sigma d\sigma}{g\tilde{\rho}^{(1+\alpha_\infty)/3}} + \frac{\alpha_*}{3}\int_1^{\tilde{\rho}} \frac{(\tilde{\alpha}-\tilde{\alpha}_\infty)d\tilde{\rho}}{\theta\tilde{\rho}^{1+(1+\alpha_\infty)/3}} + 0(\epsilon^{3/2})
\end{aligned}$$

$$(M_*^2/2)(1/\tilde{\rho}^2\sigma^2 - 1) = (\alpha_*/\epsilon)(1-\tilde{\alpha}) + 4(1 - 1/\theta) + \alpha_*(1 - \tilde{\alpha}/\theta)$$

The results of this section cannot be reduced to those in Sec. 2 by taking $\alpha_* = \epsilon K$, nor can the results of Sec. 2 be reduced to those here by taking $K = \alpha_*/\epsilon$; therefore, these two cases need to be treated separately. However, it can be shown that they do reduce to a common form when K is taken to be large in the results of Sec. 2 and α_* is taken to be small in the results here, provided that $\alpha_* = \epsilon K$ is observed. All the intermediate cases between the two, and thus all the range of K or α_* in the reaction-dominated regime, are thus covered. For this case, the equilibrium-recombination model has an error of $0(\epsilon \ln\epsilon)$ for all the flow variables θ, $\tilde{\rho}$, and $\tilde{u}$, but has a smaller error of $0(\epsilon^2 \ln\epsilon)$ in estimating the composition $\tilde{\alpha}$. Since this case corresponds to $K = 0(1/\epsilon)$, in consistency with the previous conclusion, the sudden-freezing model predicts the composition correctly with an error of $0(\epsilon)$, provided that the composition freezes far downstream. Because the dissociation level is higher in the transitional regions in this case, it can be expected that the contribution from the heat of recombination is larger. This is confirmed by the fact that in the energy integral equation of Eq. (3.9), the second integral is of $0(1)$, instead of $0(\epsilon)$ as in the case of $K = 0(1)$. Thus, the sudden-freezing model gives a poorer estimate of the temperature in the present case.

4. Comparison with Exact Numerical Solution

It is instructive to compare the solution by asymptotic expansions with the corresponding numerical solution of the exact governing equations, namely Eqs. (2.1–2.6) of Pt. I. A numerical example will also bring out actual magnitudes and features in the solution not easily discernible from the general results. For the sake of simplicity, a high-M_* and lower-α_* case is chosen so the results in Pt. I, which are explicit in σ, can be used. The example chosen is a hyperboloidal nozzle described by

$$A = A_{\text{thr}} + K_N^2 x^2 \quad (4.1)$$

which satisfies all the assumptions 1 through 8 of Pt. I for $x > 0$. The chosen equilibrium reservoir conditions are $kT_{\text{st}}/D = 0.10$ and $\alpha_{\text{st}} = 0.80$, and the chosen rate parameter in Bray's form $\phi \equiv C\rho_D(kA_{\text{thr}}/DR_2)^{1/2}/2K_N$ is 3×10^{13}. Following Bray,[2] we take $s = 0$, which leads to explicit results in the recombination region. The numerical solution for this example was furnished by G. Emanuel using a procedure developed in the work of Emanuel and Vincenti.[3]

From the equilibrium solution pertaining to the reservoir conditions and the rate parameter, we have determined the reference point from Eq. (2.8) of Pt. I by setting $Q = 1$. Associated with the reference point are the reference values α_*, T_*, u_*, ρ_*, and A_*, from which we obtain the following values for the nondimensional parameters: $\epsilon \equiv kT_*/D = 0.0535$, $\alpha_* = 0.320$, $K \equiv \alpha_*/\epsilon = 5.98$, $M_*^2 \equiv u_*^2/R_2T_* = 27.3$ ($\Gamma = 1.37$) and the corresponding value of P is 1.10. The value of K indicates that this case belongs to the reaction-dominated regime. The value of M_*^2 is high enough to justifiably use the results of Pt. I. Whereas α_* in this case does not appear to be very small, the remainders in the results of Pt. I, as we may recall, are of $0[\epsilon^{1/2}(\epsilon + \alpha_* + 1/M_*^2) + (\alpha_* + 1/M_*^2)^2]$ which is higher than $0(\alpha_*)$. In fact, the errors found in the comparison are actually smaller than the order of remainders indicates, for all flow regions and all flow variables. The hyperboloidal nozzle corresponds to $m = 1$. With $s = 0$, the asymptotic behavior of the solution far downstream belongs to type A, as discussed in Sec. 3.5 of Pt. I, for which the chemistry is known to freeze.

The comparison is presented in Fig. 1 for the dissociation fraction $\tilde{\alpha}$ vs σ, in Fig. 2 for the inverse of temperature θ vs σ and in Fig. 3 for the velocity $\tilde{u}$ vs σ. The range of σ considered extends from $\frac{1}{10}$ to 10 and beyond and presumably covers the main parts of regions (ii–iv). With $\epsilon = 0.0535$, the rapid-transition region is expected to have a spread of order $\epsilon^{1/2} = 0.232$ around $\sigma = 1$. In these figures, the analytical solutions for regions (ii–iv) are represented by thin

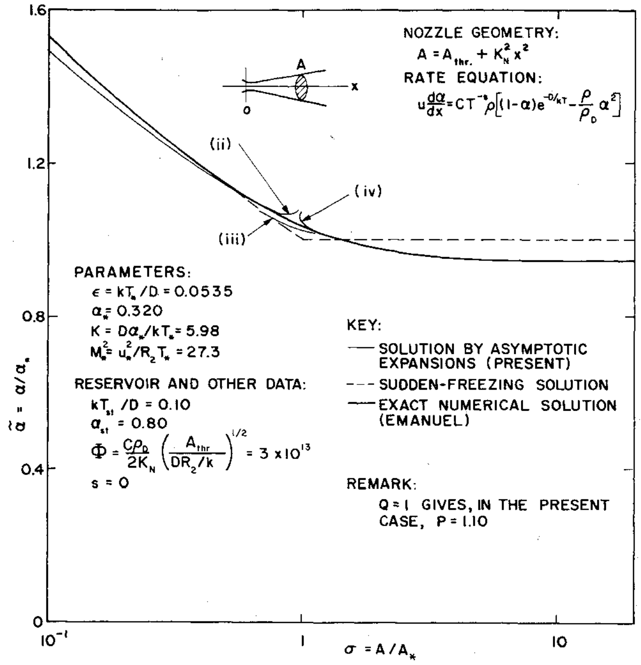

Fig. 1 Comparison of the solution of dissociation fraction α by asymptotic expansions in regime II, the solution by a sudden-freezing analysis, and the exact numerical solution for a hyperboloidal nozzle.

lines marked with their respective regions of validity and the exact numerical solution by heavy lines. Also included for reference are the results based on the sudden-freezing model which are represented by dashed lines. The analytic solutions are seen to be in good agreement with the exact numerical solution and to match with one another; the discrepancy between these curves amounts to less than 3% (which is considerably smaller than α_*^2, or 10% for the present case). Indeed throughout much of the recombination region in Figs. 1 and 2 and throughout all the regions in Fig. 3, these curves are virtually coincident.

Some features of the solution curves may be noted. In Fig. 1, the $\tilde{\alpha}$ vs σ curve appears rather straight with a negative slope in the semilog scale (as indeed predicted by both the equilibrium and the incipient-transition solutions); it curves through the rapid-transition region and flattens out finally in the recombination region. The equilibrium curve begins to depart slightly from the exact solution curve upstream of $\sigma = 1$; downstream it is not shown, but its departure from the exact solution curve becomes greater and greater. The asymptotic limit of α/α_* for $x \to \infty$ is reached well upstream of $\sigma = 10$, with the value of α frozen at about 95% of α_*. It is evident from Fig. 1 that the sudden-freezing model is adequate in this case. Although according to our theory the sudden-freezing model predicts a freezing $\tilde{\alpha}$ with an error of $\tilde{\alpha} - 1 = 0(1/K)$, the comparison shows that the error is numerically much smaller than $1/K$. In the problems analyzed numerically by Bray[2] and by Hall and Russo,[4] their values of K are comparable to the present value, and the reason for the adequacy of their sudden-freezing analyses in predicting α is therefore understandable.

The inverse temperature ratio θ shown in Fig. 2 begins on the left near unity with a small slope like $\epsilon \ln\sigma$, as indicated by the equilibrium and the incipient-transition solutions. After passing over $\sigma = 1$, it increases rapidly and departs significantly from (the unshown part of) the equilibrium solution. Thus, the transition from region (ii) to (iv) appears more rapidly in the temperature than in the dissociation fraction. Of particular significance is the difference between the curves representing the sudden-freezing analysis and the exact solution in the recombination region, confirming our conclusion that the sudden-freezing model can never correctly predict the temperature distribution. However, as anticipated, the magnitude of this difference is not excessively large. It is expected that the modified sudden-freezing temperature-formula [Eq. (4.3) of Pt. I] should yield a close enough agreement with the exact solution.

In passing, we may note in both Figs. 1 and 2 the divergent trends of our solutions for regions (ii) and (iv) away from the exact solution in the immediate vicinity of $\sigma = 1$ where they are not supposed to be valid expansions. We remark also that the composite solutions for regions (ii) and (iii) and for regions (iii) and (iv) can be carried out but have not been done.

It has been observed in Pt. I that if the chemistry freezes the velocity variation in regions (ii–iv) is small for a very high M_*^2, irrespective of K, and the sudden-freezing model yields a good approximation with an error of $0(1/M_*^2)$. This is confirmed by Fig. 3 where the sudden-freezing and exact-solution curves are very close to each other. On the other hand, over most parts of the recombination region, both curves are appreciably above $\tilde{u} = 1$. For reference, on the right of Fig. 3, we present three asymptotic values of the

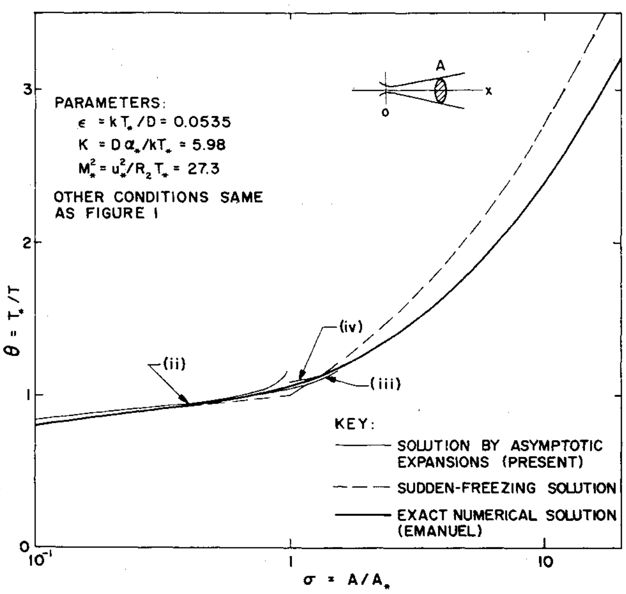

Fig. 2 Comparison of the solution of the inverse temperature ratio $\theta = T_*/T$ by asymptotic expansions in regime II, the solution by the sudden-freezing analysis, and the exact numerical solution for an hyperboloidal nozzle.

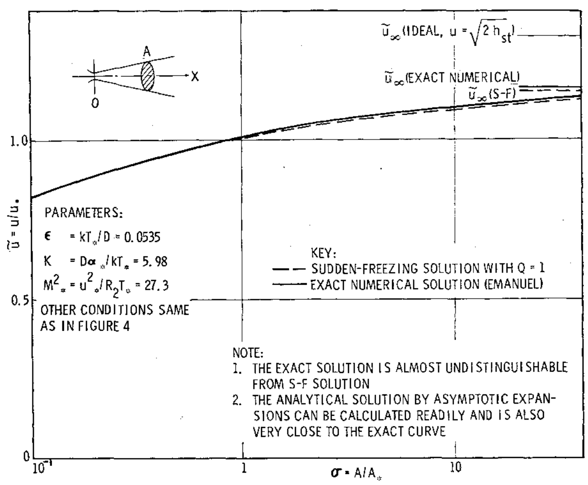

Fig. 3 Confirmation of accuracy of the sudden-freezing model with $Q = 1$ in predicting velocity distribution, hence specific impulse, for a nonequilibrium nozzle flow.

velocity ratio far downstream, corresponding, in order of increasing magnitude, to values of the sudden-freezing model, the exact solution, and the ideal maximum speed $[u = (2h_{st})^{1/2}]$. The loss in specific impulse far downstream owing to freezing is seen to be about 15% which is accurately approximated by the sudden-freezing value. Of some practical importance is, perhaps, the suggestion from this figure that the velocity approaches its terminal value much more slowly than does the dissociation fraction. Thus, the specific impulse may not be represented too well by its terminal value unless σ is extremely large.

5. Conclusions

The problem of transition from an equilibrium to a recombination flow in a nozzle for an ideal dissociating gas has been studied analytically in general in the reaction-dominated regime $[K \geq 0(1)]$ under $\epsilon \ll 1$. Other possible transitional patterns have been studied recently by the authors,[5] and by Blythe[6] in the case of vibrational nonequilibrium. The assessment of the sudden-freezing and the equilibrium-recombination models made in Pt. I is proved to hold in general in this regime. To determine the transition point meaningfully, it is shown that the most general form of Q to be used is $[M_*{}^2 - (1 + \alpha_*)]/M_*{}^2(1 + \alpha_*)$, which corresponds in all cases to $P = 1 + 0(\epsilon)$. The analytical results are found to be in good agreement with an exact numerical solution for a typical example of nonequilibrium nozzle flow.

A key restriction which remains on the present study is the equilibrium-throat assumption, which places a lower limitation on $M_*{}^2$. To study the nonequilibrium-throat effect we may have to introduce two more distinct regions to handle the singular behavior of the solution near the reservoir and the critical point. An alternative approach is to consider the inverse problem of a nozzle with a prescribed pressure distribution[7] or other flow-property distributions, instead of a prescribed area variation. In these inverse problems the critical point does not exist (the flow rate has to be given) and thus one less distinct region is needed. Perhaps, the use of pressure as the independent variable will make the present study more meaningful to flowfield and wake studies. Certainly lacking still is a general study on the problem in the weak-interaction regime $[K \leq 0(\epsilon)]$, of which Blythe's previous work[8] is a special case. Another interesting aspect that remains to be explored more in depth is the asymptotic behavior of the solution for a divergent nozzle far downstream. A qualitative study of the topology of the integral curves at the downstream infinity should provide more definitive information than given here.

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